Math 215, Winter 2014
Midterm 3, March 27

Name: **SID:**

Instructor: **Section:**

Instructions

- The total time allowed is 60 minutes.
- The total score is 50 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Calculators, phones and cheat sheets are not allowed.

Problem	**Points**	**Score**
1	16	
2	12	
3	12	
4	10	
TOTAL	50	

1. (16 marks)

a The general solution to $\frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Find the general solution to

$$\frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

b If $F(s) = \frac{se^{-2s}}{s^2+2s+5}$ is the Laplace Transform of $f(t)$, determine $f(t)$.

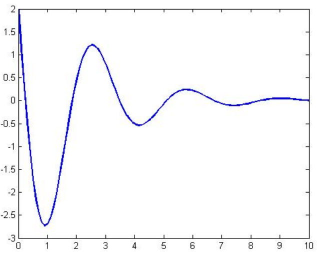
c Compute the Laplace transform of $f(t) = \begin{cases} 0, t \leq 3 \\ \sin(\pi t) \text{ otherwise.} \end{cases}$

2. **Matching** (12 marks: 1 mark each)

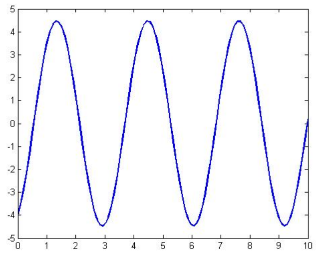
In each of the parts *i-iv* below, you are given three constant coefficient linear differential equations with *unspecified* initial conditions. In each grouping, a differential equation matches to one of the options provided.

Solution (circle the correct answer in each case):

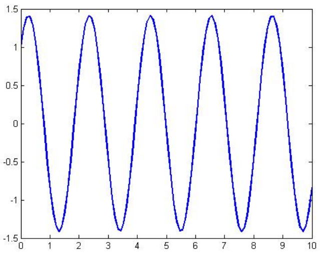
(i)	$y'' + 4y = 0$	matches plot	A	B	C
	$y'' + y' + 4y = 0$	matches plot	A	B	C
	$y'' + 9y = 0$	matches plot	A	B	C
(ii)	$y'' + y' + y = 0$	matches plot	D	E	F
	$y'' + y' + 2y = 0$	matches plot	D	E	F
	$y'' + y' + \frac{1}{4}y = 0$	matches plot	D	E	F
(iii)	$y'' + 4y' + 8y = 0$	matches plot	G	H	I
	$3y'' + 4y' + 8y = 0$	matches plot	G	H	I
	$3y'' + 4y' + \frac{8}{3}y = 0$	matches plot	G	H	I
(iv)	$y'' + y = \sin t$	matches plot	J	K	L
	$y' + y' + 4y = \sin t$	matches plot	J	K	L
	$y'' + 4y = \sin t$	matches plot	J	K	L



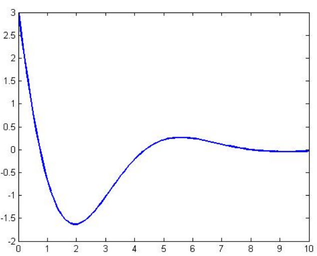
(A)



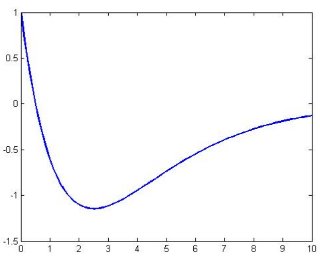
(B)



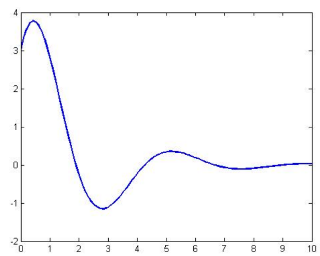
(C)



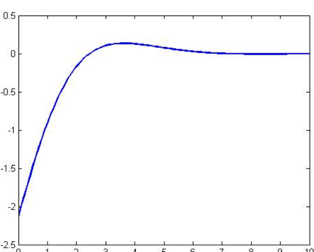
(D)



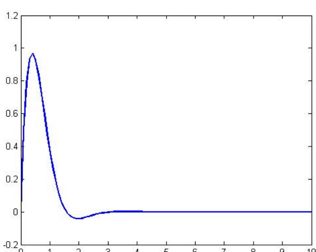
(E)



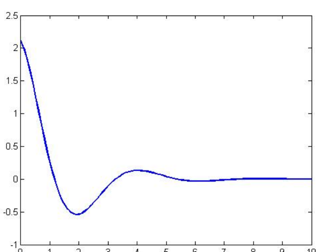
(F)



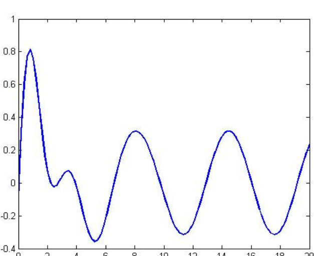
(G)



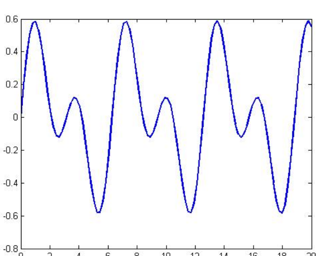
(H)



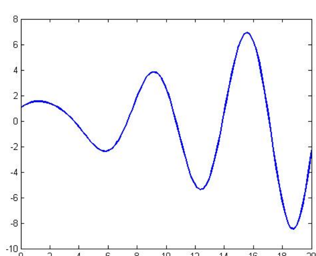
(I)



(J)



(K)



(L)

3. A series LRC circuit is driven by an AC voltage. The charge Q in Coulombs on the capacitor satisfies

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = V_0 \sin(\omega t)$$

where $L = 1$ H, $R = 5\Omega$, $C = \frac{1}{4}$ F, $V_0 = 2$ V, and $\omega = 2$ Hz.

Solve for $Q(t)$ with $Q(0) = Q'(0) = 0$.

4. A 1-kg mass on a spring undergoes undamped, free horizontal motion along the $x-$axis with an oscillation period of 2 s. Let $x(t)$ be the displacement of the mass relative to the equilibrium position $x = 0$. At $t = 3$ s, the mass is at position $x = 1$ m with velocity $dx/dt = 4$ m/s.

a. Write down a differential equation for the position $x(t)$. Your equation should include an unknown spring constant k.

b. Solve for $x(t)$ in terms of k.

c. Determine k.

d. What is the amplitude of $x(t)$?

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(5)
$f'(t)$	$sF(s) - f(0)$	(6)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	(7)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(8)
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(9)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(10)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(11)
e^{at}	$\dfrac{1}{s-a}$	(12)

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
te^{at}	$\dfrac{1}{(s-a)^2}$	(13)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(14)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(15)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(16)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(17)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(18)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(19)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(20)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(21)
$t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$	(22)

Trig identities

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B
\end{aligned}
$$