Math: the art in problem solving

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Math is everywhere

- quantifiable descriptions
- glacier motion, spreading of diseases, market inflation, digitally recorded music, ...
- real world $\to$ model $\to$ abstractify $\to$ analyze $\to$ interpret
- math is useful

Outline

- applied math technique intros
- electrodialysis application
- fusion application
- mass spectrometry application
- concluding remarks

Modelling Overview

- “sketch” relevant physics, biology, etc.
- assign variables, write equations
- insightful results

Modelling Example

- finding the dimensions of a cylindrical tank with open top having a volume of 1000 cm^3 with least possible surface area
- let $h =$height, $r =$radius; need $\pi r^2 h = 1000$ with minimum value of $2\pi rh + \pi r^2$

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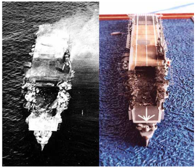
- cheaper than building thousands of cylinders!

Nondimensionalization Overview



- make "diorama"
- remove dimensions, reduce constants
- key relationships
- improved numerics
- all variables comparable: velocity, pressure, time, etc.

Nondimensionalization Example

- rabbit population P has carrying capacity C satisfies $P'(T) = rP(1 - P/C)$ at time T, initial population $P(0) = P_0$
- units: $[P] = [C] = [P_0] =$ rabbits, $[r] = 1/$time, $[T] =$time
- nondimensionalize to

$$p'(t) = p(1-p), \quad p(0) = p_0$$

- qualitative behaviour: p_0

Asymptotics Overview

- mathematical "impressionism"
- exploit relative smallness of parameter(s)
- approximate model problem solution accurately

$$\text{solution} = \text{estimate} + \text{correction} + \text{smaller correction} + \ldots$$

- analytic insight

Asymptotics Example

- consider $0.0001x^3 - x + 1 = 0$; let $\epsilon = 0.0001 \ll 1$
- asymptotic prediction:

$$x \sim 1 + \epsilon + \ldots$$
$$x \sim \frac{1}{\sqrt{\epsilon}} - \frac{1}{2} + \ldots$$
$$x \sim \frac{-1}{\sqrt{\epsilon}} - \frac{1}{2} + \ldots$$

Asymptotic	Exact
1.0001...	1.0001...
99.5...	99.496...
-100.5...	-100.496...

Numerical Analysis Overview

- “digital snapshot”
- arbitrarily high precision
- control error terms
- may require high computational power

Numerical Analysis Example

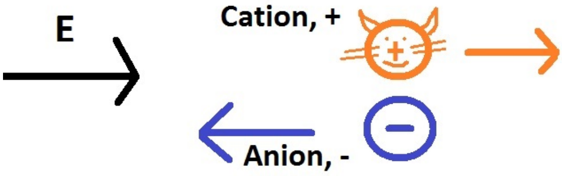
- evaluate $I = \int_0^1 \sqrt{x^5 + 1} dx$
- Riemann sum: let $N > 0$, $h = 1/N$ and $x_j = jh$ so that $I = h \sum_{j=1}^{N} \sqrt{x_j^5 + 1} + \text{error}$
- theory: $|\text{error}| \leq \frac{\text{some constant}}{N} \to 0$ as $N \to \infty$

N	Sum
10	0.999
100	1.066
1000	1.0738
10000	1.0746
100000	1.0747

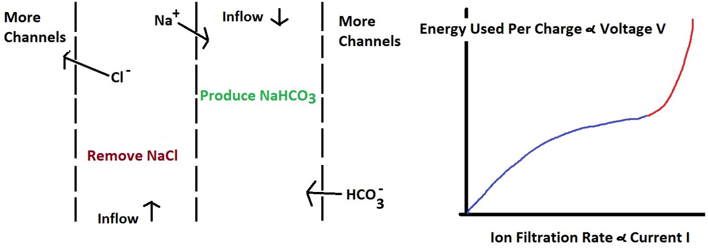
- better methods exist!

Electrodialysis Motivation

- clean water and CO_2 sequestration good
- many pollutants ionic
- electrodialysis: filter out impure ions from water, make useful products with electric field and selective membranes

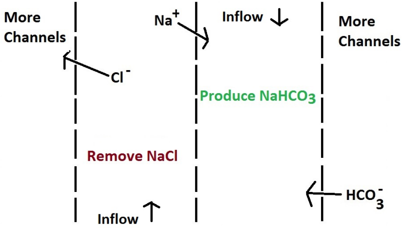


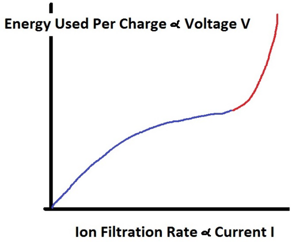
Problem



- UBC chemical engineers building prototypes, operational conditions important

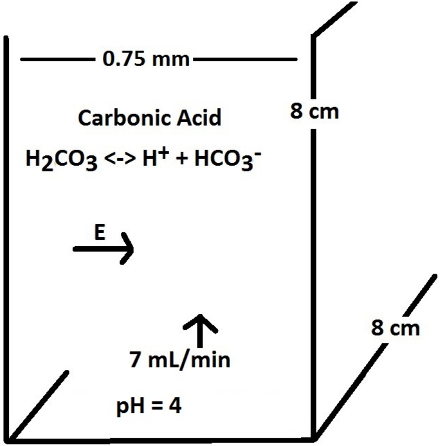
Problem





- UBC chemical engineers building prototypes, operational conditions important
- *Given experimental data of input concentrations, the total voltage and the total current drawn across the series of channels, what losses can be attributed to each individual channel and what is taking place in each channel?*

Carbonic Acid Channel



- given voltage drop V, determine the current I

Models

- model 0: uniform $[H^+]$ and $[HCO_3^-]$, current driven by voltage gradient - wrong species cross membranes

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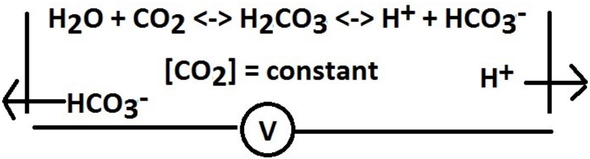
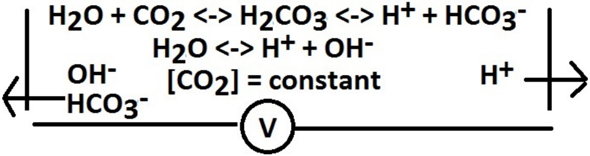
- model 0: uniform $[H^+]$ and $[HCO_3^-]$, current driven by voltage gradient - wrong species cross membranes
- model 1: spatially varying $[H^+]$ and $[HCO_3^-]$ without reactions - too few ions
- model 2: with reactions

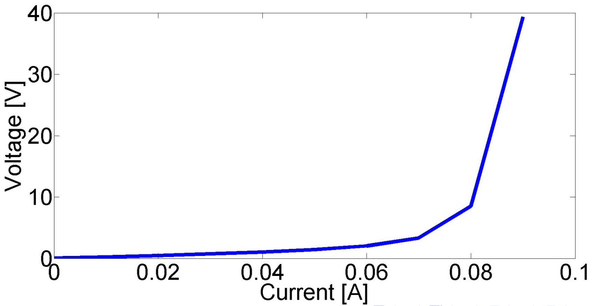


but ions not produced fast enough #fail

Models

- model 3





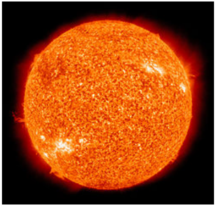
Models

- numerics on asymptotic model more consistent with experimental data
- discovered device doesn't operate the way intended!

Models

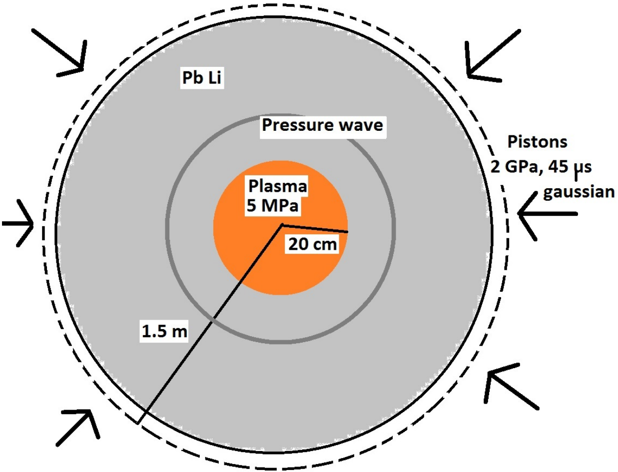
- numerics on asymptotic model more consistent with experimental data
- discovered device doesn't operate the way intended!
- "... all models are wrong, but some are useful." - George Box

Fusion Motivation

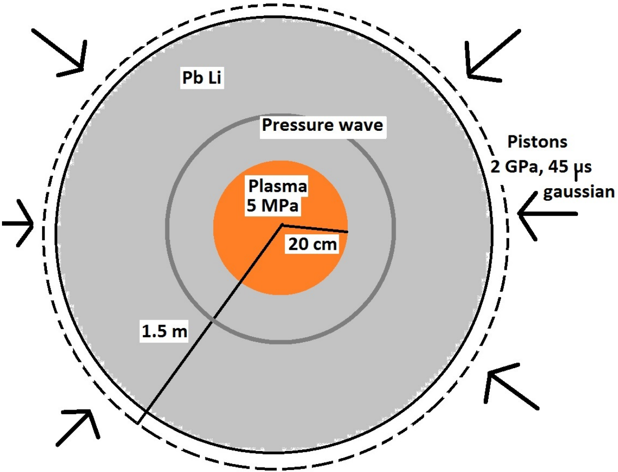


- fusion energy in sun
- ${}^{3}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \to {}^{4}_{2}\mathrm{He} + {}^{1}_{0}\mathrm{n} + 17.6$ MeV and other reactions
- clean energy
- sustain high particle density and temperature
- fusion on earth?
- magnetized target fusion at General Fusion
- design: compress plasma in imploding shell of lead-lithium, driven by fast high-pressure impulses

Problem


Pb Li
Pressure wave
Pistons
2 GPa, 45 μs
gaussian
Plasma
5 MPa
20 cm
1.5 m

Problem



- *How much will the plasma compress under such a design?*

Simple Dimensionless Model

- spherical symmetry
- homogeneous plasma
- nonlinear, coupled system of PDEs: mass and momentum conservation
- free boundaries - need to solve for radii within problem
- nondimensionalization

  small initial plasma radius
  very small initial density disturbance
  very small impulse time
  very, very small initial plasma pressure

Insights Obtained

- wave-like behaviour: sound speed dominates
- pressure amplitude grows like $1/r$
- almost all input energy reflected!
- fusion may be within reach - Lawson condition

Insights Obtained

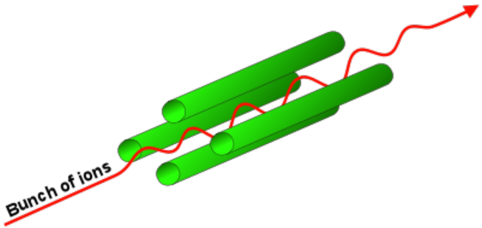
- minimum radius R_L^*:

$$R_L^* \approx \frac{C_s^4 P_{\text{plasma},0} R_{\text{inner},0}^7 \varrho_0^3}{\pi P_{\max}^4 R_{\text{outer},0}^4 T_0^2} = 4\text{mm}$$

- agrees well with numerical methods

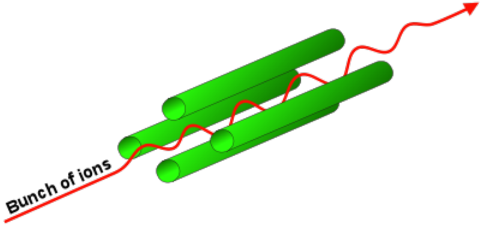
Parameter	Meaning
C_s	lead sound speed at atmospheric pressure
$P_{\text{plasma},0}$	initial plasma pressure
$R_{\text{inner},0}$	initial plasma radius
ϱ_0	lead density at atmospheric pressure
$P_{\max}$	piston impulse pressure
$R_{\text{outer},0}$	initial lead radius
T_0	impulse time scale

Spectrometry Motivation and Problem



- detect chemical species/isotopes with mass
- charge-to-mass separation in mass spectrometer
- some current means
  - magnetic fields costly
  - time of flight discrete
  - quadrupole too specific

Spectrometry Motivation and Problem



- detect chemical species/isotopes with mass
- charge-to-mass separation in mass spectrometer
- some current means
  - magnetic fields costly
  - time of flight discrete
  - quadrupole too specific
- *PerkinElmer: come up with a design for a mass-spectrometer without magnetic fields, with continuous measurement and which can detect any mass.*

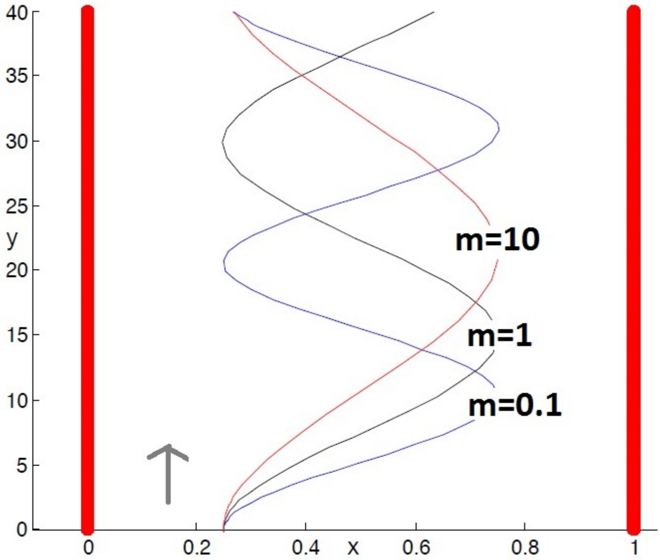
Concepts

- acceleration in electric field proportional to charge-to-mass ratio and field strength
- positive charge oscillates between fixed positive charges (one-dimensional)
- same amplitude and different masses, different periods

Idea

- fire ion beam into tube of charges
- predict qualitative trajectories via asymptotics
- oscillation wavelength mass-dependent
- numerics of ODEs verify distinct masses spatially separate

Results



Art

- each technique has role
- combination often useful
- results extremely beautiful/satisfying

Final Remarks

- thanks to many collaborators on these projects:
  - Arman Bonakdarpour, Saad Dara, and David Wilkinson (UBC Chemical and Biological Engineering Department)
  - Sandra Barsky and Aaron Froese (General Fusion)
  - Iain Moyles, Michael Ward, and Brian Wetton (UBC Math Department)
  - Samad Bazargan (PerkinElmer)
  - others
- some suggested applied courses:
  - MATH 215, 316, 400, 401 (differential equations)
  - MATH 361 (modelling, math-bio)
  - MATH 210, 405 (numerics)
  - MATH 450 (asymptotics)
  - some non-math science courses!
- slides posted at: math.ubc.ca/∼MLRTLM/#talks