Magnetized Target Fusion: Insights from Mathematical Modelling

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Context

A local Canadian research company is designing a reactor to produce fusion energy, which has never been achieved before. Success with this could hold immense promise for clean and sustainable energy sources in the future. In this talk, we will look into the mathematical analysis used to yield insights into the design work. Key components:
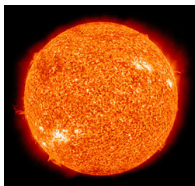
- **Reactor model**
- **Numerical finite volume approach**
- **Formal asymptotic approach**

Thanks to...

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- Sandra Barsky and Aaron Froese, General Fusion
- George Bluman, Michael Ward, and Brian Wetton, UBC Math Department
- Randy LeVeque, University of Washington Applied Mathematics Department
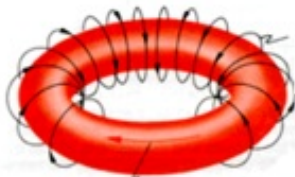- My colleagues and fellow graduate students

Fusion

- Fusing atomic nuclei yield new nuclei plus energy
- Occurs naturally in sun: pressure $\sim$ 300 G atm, temperature $\sim 10^7$ K
- Potential clean energy source
- Tritium-deuterium plasma reaction:
  $^3_1$ H $+^2_1$ H $\rightarrow ^4_2$ He $+$ n $+$ 17.6 MeV of *energy*
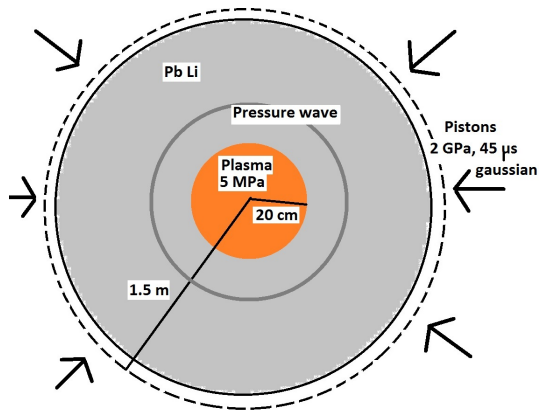- Depends on temperature, thermal distribution, etc.

Challenges

- Immense temperature and pressure must be sustained
- Thermal and radiative losses
- Lawson criterion [1] for energy yield:
  density $\times$ temperature $\times$ time $\geq 4 \times 10^{15}$ cm^{-3} KeV s

General Fusion design

- General Fusion founded by Michel Laberge in 2002
- Design magnetized target fusion reactor
- Isolate spheromak plasma with magnetic field
- Crush plasma in imploding metal cavity

General Fusion design

- How feasible is such a design?

Lead-Lithium

- Euler equations for mass and momentum conservation
- With density ρ, velocity v, and pressure P:

$$\rho_t + \nabla \cdot (\rho v) = 0 \qquad \text{(mass)}$$
$$(\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla P = 0 \qquad \text{(momentum)}$$
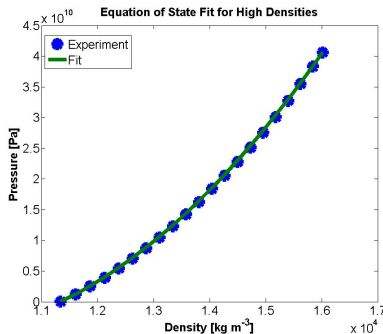$$P = P(\rho) \qquad \text{(equation of state)}$$

- Nonlinear coupled system of PDEs

Lead-Lithium

- Quadratic fit to high-pressure Lead experiments [2]

Piston and plasma pressure

- Gaussian piston pressure:

$$P(t) = P_{\text{atmospheric}} + (P_{\text{impact}} - P_{\text{atmospheric}})e^{-t^2/t_0^2}\theta(t)$$

- Plasma has gas and magnetic pressure
- Ideal gas of volume V:

$$P_{\text{gas}} \propto V^{-\gamma}, \quad \gamma \geq 1$$

- Magnetic pressure [3]:

$$P_{\text{magnetic}} \propto R_{\text{major}}^{-4}$$

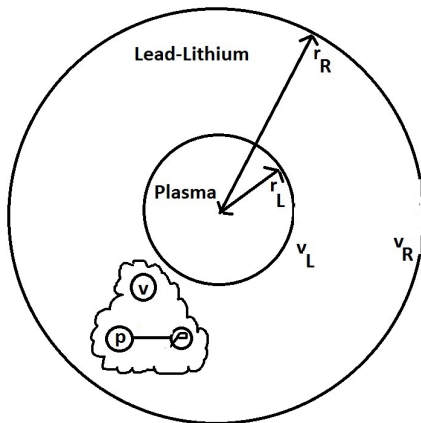Michael Lindstrom

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Assumptions and simplifications

- Spherical symmetry
- Isentropic (reversible) conditions so $\gamma = 5/3$
- Initially $P_{\text{gas}} \approx 0.1 P_{\text{magnetic}}$
- Initial plasma temperature 100 eV
- System starts in equilibrium
- No mixing implies free boundary problem at plasma-metal and metal-piston interfaces:

$$\frac{\mathrm{d}}{\mathrm{d}t} r_{\text{boundary}}(t) = v(r_{\text{boundary}}(t), t)$$

- Neglect thermal and radiative energy losses

Overall model

Overall model

In $r_L(t) < r < r_R(t)$, $t > 0$, dimensionless system has form:

$$\rho_t + \frac{1}{r^2}(r^2 \rho v)_r = 0, \quad (\rho v)_t + p_r + \frac{1}{r^2}(r^2 \rho v^2)_r = 0$$

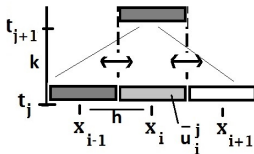$$p = p(\rho), \quad \frac{\mathrm{d}r_{L,R}}{\mathrm{d}t} = v(r_{L,R}(t), t)$$

$$p(r_L(t), t) = p_L(r_L(t)), \quad p(r_R(t), t) = f(t)$$

$$v(r, -\infty) = 0, \quad p(r, -\infty) = p_L(r_L(-\infty))$$

$$r_L(-\infty) = \chi \epsilon^{1/2}, \quad r_R(-\infty) = 1$$

First-order finite volumes

- Given $u_t + (f(u))_x = 0$ denote $\bar{u}(x_i, t) = \frac{1}{h} \int_{x_i - h/2}^{x_i + h/2} u(x, t) \mathrm{d}x$

- $\frac{\mathrm{d}\bar{u}(x_i, t)}{\mathrm{d}t} = \frac{1}{h} \int_{x_i - h/2}^{x_i + h/2} u_t(x, t) \mathrm{d}x = -\frac{f(u(x_{i+1/2}, t)) - f(u(x_{i-1/2}, t))}{h}$

- Choose $\mathcal{F}^{\mathcal{L}}(u_{i-1}, u_i) = f(u_{i-1})$ or $f(u_i)$ depending on f'

- Stable time-step $k < h/\sup |f'|$,
  $u_i^{j+1} = u_i^j - \frac{k}{h}(\mathcal{F}^{\mathcal{L}}(u_i^j, u_{i+1}^j) - \mathcal{F}^{\mathcal{L}}(u_{i-1}^j, u_i^j))$

Second-order flux-limited finite volume idea

- Smoothness: $u_t + f'(u)u_x = 0$
- Use $u_{tt} = -f''(u)u_t u_x - f'(u)u_{tx} = (f'(u)^2)_x u_x + f'(u)^2 u_{xx}$
- Lax-Wendroff:
  $u(x, t+k) = u(x, t) + ku_t(x, t) + \frac{k^2}{2}u_{tt}(x, t) + O(k^3)$
- Corresponding high-resolution flux $\mathcal{F}^{\mathcal{H}}$
- Flux limited: take flux $\mathcal{F} = \phi\mathcal{F}^{\mathcal{L}} + (1 - \phi)\mathcal{F}^{\mathcal{H}}$
- Limiter ϕ largest where derivative $|u'(x)|$ largest

Convergence

- Numerical solution $u_{\text{num}}(x, t)$ and exact solution $u_{\text{ex}}(x, t)$:

$$E = \int |u_{\text{num}}(x, t) - u_{\text{ex}}(x, t)| dx \to 0, \quad \text{as } N \to \infty$$

- Near shocks, $E = O(h)$
- Near smooth regions $E = O(h^n)$ for n^{th}-order method

Coordinate transformation

- Transform from moving to fixed computational domain
- Let $\tau = t$ and define $\Delta(\tau) = r_R(\tau) - r_L(\tau)$,
  $\Gamma(\tau) = v_R(\tau) - v_L(\tau)$
- Set $y = \frac{r - r_L(\tau)}{\Delta(\tau)}$, $y \in [0, 1]$
- $p(r_R(t), t) = p(1, \tau)$, etc.
- New conservation laws e.g. mass:

$$\rho_\tau + \{\frac{1}{\Delta}[-(v_L + \Gamma y)\rho + \rho v]\}_y = \frac{-2}{r_L + \Delta y} - \frac{\Gamma}{\Delta}\rho$$

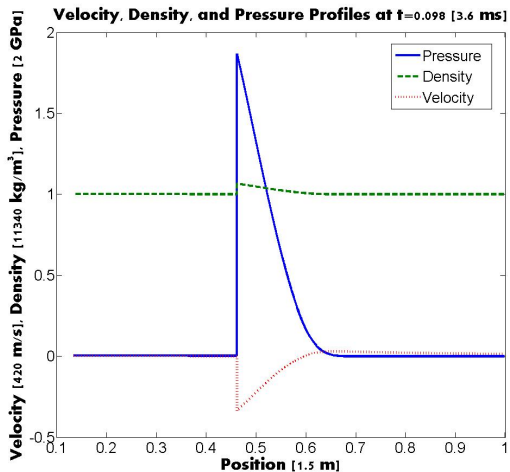Michael Lindstrom

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Finite volume overview

- Pseudocode [4]:
  - eigen analysis for system
  - constant extrapolation to ghost-points
  - use projections for limiters
  - homogeneous Riemann problem then use source
  - update time and boundary data
- Verify L^1-convergence

Table: Convergence of Numerical Scheme at fixed time

N	Velocity Error	Density Error	Mass Error
1000	5.68×10^{-4}	1.11×10^{-3}	4.79×10^{-4}
4000	1.39×10^{-4}	2.77×10^{-4}	1.20×10^{-4}
16000	3.51×10^{-5}	6.93×10^{-5}	2.99×10^{-5}
Rate	1.01	1.00	1.00

Pulse profiles

Abridged sensitivity analysis

Table: Min radius R_{min}, Lawson triple product Π_L, impact pressure P_{impact}, initial plasma radius $R_{plasma,0}$, initial sphere radius $R_{lead,0}$, time scale t_0.

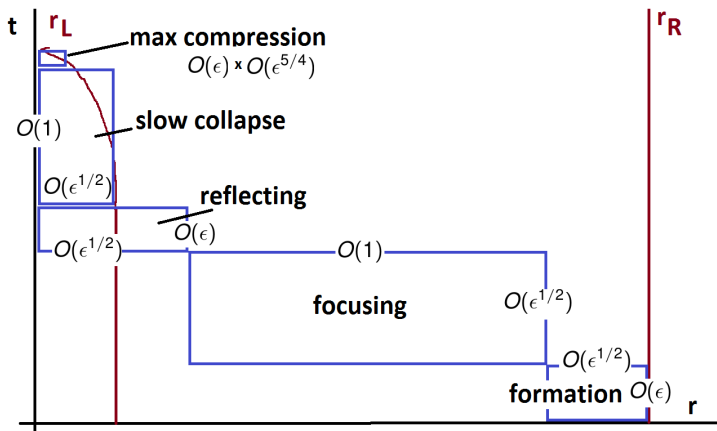| System	R_{min} (cm)	Π_L (10^{15} keV s cm^{-3})
Baseline	3.6	0.52
$R_{plasma,0} \times 1.1$	5.5	0.25
$P_{impact} \times 1.1$	3.0	0.64
$R_{lead,0} \times 1.1$	3.0	0.92
$t_0 \times 1.1$	3.1	0.61
$P_{impact} \times 2$	1.2	16
$R_{lead,0} \times 2$	0.84	2.5
$t_0 \times 2$	1.3	1.9

Big picture

- Modifications:
    - Gaussian impulse pressure without heaviside
    - linearized $P(\rho)$ equation of state
    - only magnetic pressure
- Relevant scales:
    - asymptotic parameter $\epsilon \ll 1$
    - big sound speed: $b\epsilon^{-1/2}$
    - impulse pressure: $O(1)$
    - small initial plasma radius: $\chi\epsilon^{1/2}$
    - very small impulse time: $O(\epsilon)$
    - very, very small initial pressure: $\mu\epsilon^{3/2}$
- Matched asymptotics [5]

Asymptotic regimes

Pulse formation and focusing

- Formation:
  - $\rho(y,\tau) \sim 1 + \epsilon\rho_1 + \epsilon^{3/2}\rho_2, \quad v(y,\tau) \sim \epsilon^{1/2}v_0 + \epsilon v_1$
  - Riemann Invariants [6]- plane wave solutions in $(-\infty, 0] \times [0, \infty)$:

$$\rho_1(y,\tau) = \frac{1}{b^2}e^{-(\tau+y/b)^2}, \qquad v_0(y,\tau) = \frac{-1}{b}e^{-(\tau+y/b)^2}$$

  - need divergent ρ_2, v_1 for matching
- Focusing:
  - amplitude growth:

$$\rho \sim 1 + \epsilon\rho_1 + \epsilon^{3/2}\rho_2, \quad v \sim \epsilon^{1/2}v_0 + \epsilon v_1 \quad \text{(outer)}$$

$$\rho \sim 1 + \epsilon^{1/2}\rho_1 + \epsilon\rho_2, \quad v \sim v_0 + \epsilon^{1/2}v_1 \quad \text{(inner)}$$

  - linear acoustic limit [7]:

$$\rho_{1,\hat{t}} + v_{0,\sigma} + \frac{2}{\sigma}v_0 = 0, \qquad v_{0,\hat{t}} + b^2\rho_{1,\sigma} = 0$$

Pulse reflection

- Complete reflection of ρ_1 and v_0, negligible compression
- Next order:

$$v_{1,\hat{t}} + b^2 \rho_{2,\sigma} = -(\rho_1 v_0)_{\hat{t}} - (v_0^2)_\sigma - \frac{2}{\sigma} v_0^2$$

- Careful integration and boundary condition trickery gives

$$v_1(\sigma, \infty) = \frac{-\sqrt{2\pi}}{b^2 \chi \sigma^2}$$

- Residual velocity field $v \sim \epsilon^{1/2} v_1$ compresses plasma!
- Rayleigh energy argument [8]or more dynamics

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Plasma compression

- Integration of PDEs [9]with reformulation gives free boundary ODE
- Initially
  - $\rho \sim 1 + \epsilon^2 \hat{\rho}$, $v \sim \epsilon^{1/2} \hat{v}$
  - $O(\epsilon^{1/2})$ space scale, $O(1)$ time scale
  - find $\hat{v}(\sigma_L) = \hat{v}_L = \frac{-\sqrt{2\pi}}{b^2 \chi^{3/2} \sigma_L^{3/2}}$; inner wall at σ_L
- Finally
  - $\rho \sim 1 + \epsilon^{1/2} \hat{\rho}$, $v \sim \epsilon^{-1/4} \hat{v}$
  - $O(\epsilon)$ space scale, $O(\epsilon^{5/4})$ time scale
  - find $\hat{v}(z_L) = \hat{v}_L = \frac{-\sqrt{2Az_L - 2\mu}}{z_L^2}$; inner wall at z_L
- Matching gives A
- Turnaround point when $\hat{v}_L = 0$

Minimum radius

- Minimum radius:

$$r_{\min} \sim \frac{b^4 \chi^3 \mu}{\pi} \epsilon$$

- Agreement with numerics as $\epsilon \downarrow 0$ with
  $b = 1.05, \chi = 0.937, \mu = \pi$:

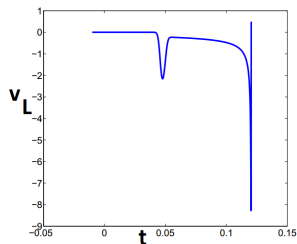| ϵ	$|r_{\min,\text{num}} - r_{\min,\text{asy}}|$
0.02	0.00845
0.01	0.00400
0.005	0.00056
0.0025	0.00008

Key insights

- Dimensional minimum radius

$$R_{\min} \approx \frac{C_s^4 P_{\text{plasma},0} R_{\text{plasma},0}^7 \varrho_0^3}{\pi P_{\text{impact}}^4 R_{\text{lead},0}^4 t_0^2} = 1.6 \text{ cm}$$

Symbol	Meaning	Symbol	Meaning
C_s	lead sound speed	$P_{\text{plasma},0}$	initial pressure
$R_{\text{plasma},0}$	initial plasma radius	ϱ_0	lead density
P_{impact}	piston pressure	$R_{\text{lead},0}$	initial lead radius
t_0	impulse time scale		

- Qualitative consistency with numerics

Key insights

- consistent compression profile
- wave-like behaviour: sound speed dominates
- almost all input energy reflected:

$$E_{\text{input}} \sim \frac{\sqrt{8\pi^3}}{b}\epsilon^{3/2}, \quad E_{\text{compression}} \sim \frac{4\pi^2}{b^4\chi^3}\epsilon^{5/2}$$

Results and future work

- Results:
    - numerics and asymptotics consistent
    - sensitivity to parameters, energy yield may be within reach
    - larger outer sphere radius and impact pressure
- Future directions:
    - incorporate more plasma physics
    - consider angular instabilities
    - more precise assessment of design

[1] Lawson, Some Criteria for a Power..., Proc. Phys Society B, 1957.

[2] Rothman et al., Measurement of principal isentropes..., J. Phys D, 2005.

[3] Woodruff et al., Adiabatic compression of a doublet field..., JOFE, 2008.

[4] LeVeque, Finite Volume Methods for Hyperbolic Problems, 2004.

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[6] Whitham, Linear and Nonlinear Waves, 1974.

[7] Landau and Lifschitz, Fluid Mechanics, 1959.

[8] Lord Rayleigh, On the Pressure Developed..., Phil. Mag. Series 6, 1917.

[9] Brennan, Cavitation and Bubble Dynamics, 1995.